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Extended objects in quantum systems and soliton solutions

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It is shown that the soliton solutions of the classical Euler equations are closely related to the extended objects created in quantum many-body systems. The general argument is supplemented by a concrete example which shows how the boson transformation applied to a quantum system leads to the static soliton solution in the $(1+1)$-dimensional $\lambda\phi^4$ model.

There has recently been an active interest in the study of extended objects which are described by the soliton solutions of the Euler equations for classical fields (say ϕ).[1] The Ginzburg-Landau equation[2] can also be regarded as a kind of Euler equation in which ϕ is the order parameter. In parallel to the study of soliton solutions of the Euler equations, we have been formulating a systematic method for the construction of extended objects which appear in quantum many-body systems.[3] This method begins with a set of Heisenberg equations for quantized fields (say ψ) and makes use of the so-called boson transformation method[4] to obtain those states in which classically behaving extended objects and quantum modes coexist. This method has been applied to the study of extended objects in various ordered states such as dislocations, grain boundaries, point defects and surface sound waves in crystals[5] and vortices, surface electromagnetic waves, Josephson currents in superconductors, etc.[4,6]

The aim of this paper is to present a brief description of the relation between the above-mentioned two approaches in the study of extended objects. In particular, it will be pointed out that the extended objects constructed by the boson method become the soliton solutions of the Euler equations when the Planck constant, h, is ignored, implying that the soliton solutions can be regarded as the extended objects with quantum origin. We feel that this consideration sheds light on the question about the gap between the classical and quantum physics.

Let us begin with a brief summary of the boson method. Consider a Heisenberg equation

$$\Lambda(\partial)\psi = F[\psi]. \tag{1}$$

To make the essence of the boson method transparent, we first consider a simple case, in which no composite particles appear and the perturbative expansion method is usable. We assume also that ψ is a scalar field. Equation (1) leads to the Yang-Feldman equation

$$\psi = \varphi^{\text{in}} + [\Lambda(\partial)]^{-1}F[\psi], \tag{2}$$

where φ^{in} is a free boson field satisfying

$$\Lambda(\partial)\varphi^{\text{in}} = 0. \tag{3}$$

It is important to note here that the mass in the $\Lambda(\partial)$ is the physical mass and, therefore, that φ^{in} is renormalized (or physical) free field. To understand this, we rewrite (2) in the form

$$\langle a|\psi(x)|b\rangle = \langle a|\varphi^{\text{in}}|b\rangle + (-i)\int d^4y\,\Delta(x-y)\langle a|F_y[\psi]|b\rangle, \tag{4}$$

where the states $|a\rangle$ and $|b\rangle$ are the ones in the Fock space of φ^{in}, and $F_y[\psi]$ means $F[\psi]$ at the space-time position y. The function $\Delta(x)$ in (4) is the Green's function

$$\Lambda(\partial)\Delta(x) = i\delta(x). \tag{5}$$

The second term on the right-hand side of (4) diverges unless $\langle a|F_y[\psi]|b\rangle$ vanishes in a reasonable manner at the limit $t_y \to \pm\infty$. To avoid this divergence we must choose φ^{in} to be the renormalized one. The above argument implies that we regard the Heisenberg equation as an equation among matrix elements (i.e., the weak relation). All the equations for Heisenberg operators in the following should be understood as weak relations.

When we solve (2) by successive iteration, we are led to the usual perturbative expansion. The result is an expression of ψ in terms of φ^{in},

$$\psi(x) = \psi[x:\varphi^{\text{in}}], \tag{6}$$

which is called the dynamical map.[7]

Let us now introduce a c-number function $f(x)$ which satisfies

$$\Lambda(\partial)f = 0. \tag{7}$$

Then we can generalize the Yang-Feldman equation as

$$\psi = \varphi^{\text{in}} + f + [\Lambda(\partial)]^{-1}F[\psi], \tag{8}$$

which obviously satisfies (1). Solving (8) by successive iteration, we obtain a new solution of (1),

$$\psi^f(x)=\psi(x:\varphi^{\text{in}}+f). \tag{9}$$

Note that ψ^f is equal to ψ in which φ^{in} is replaced by $\varphi^{\text{in}}+f$ (the boson transformation). The fact that both ψ and ψ^f satisfy the same Heisenberg equation (1) is the content of the boson transformation theorem,

$$\Lambda(\partial)\psi^f=F[\psi^f]. \tag{10}$$

Let us now introduce the c-number field ϕ^f as

$$\phi^f=\langle 0|\psi^f|0\rangle. \tag{11}$$

When we ignore h (the Planck constant), ϕ^f will be denoted by ϕ_0^f,

$$\phi_0^f=\lim_{h\to 0}\phi^f. \tag{12}$$

Note that now the difference between the vacuum expectation value of the product of ψ^f and the product of the vacuum expectation value of ψ^f is due to the contraction of in-fields, which creates the loop diagrams in the course of successive iteration applied to (2). Since the contraction of in-fields creates terms which vanish at $h=0$, we can write

$$\langle 0|F[\psi^f]|0\rangle=F[\phi^f]+O(h), \tag{13}$$

where $O(h)$ stands for those terms which vanish at $h=0$. Thus (8) leads to

$$\phi_0^f=f+[\Lambda(\partial)]^{-1}F[\phi_0^f], \tag{14}$$

which gives the classical Euler equation

$$\Lambda(\partial)\,\phi_0^f=F[\phi_0^f]. \tag{15}$$

The above argument shows that ϕ_0^f is given by tree diagrams only.

Summarizing, when we calculate $\psi(x:\varphi^{\text{in}})$ by means of the tree approximation, the vacuum expectation value of the boson-transformed one (i.e., $\langle 0|\psi^f|0\rangle$) gives ϕ_0^f which satisfies the classical Euler equation.

Let us now illustrate the construction of the soliton solution ϕ_0^f from the boson transformation by using the following model of $(1+1)$ dimensions:

$$(-\partial^2-\mu^2)\psi(x)=\lambda\psi^3(x). \tag{16}$$

Here $x=(x_0,x_1)$ and $\psi(x)$ is a scalar Heisenberg field. We are particularly interested in finding what choice of the boson transformation function $f(x)$ leads to the static soliton solution.

Using the notation

$$v\equiv\langle 0|\psi(x)|0\rangle, \tag{17}$$

we define the Heisenberg operator $\rho(x)$ by the relation

$$\psi(x)=v+\rho(x). \tag{18}$$

Equation (16) then leads to

$$(-\partial^2-m^2)\rho(x)=\tfrac{3}{2}mg\rho^2(x)+\tfrac{1}{2}g^2\rho^3(x), \tag{19}$$

and also to $\lambda v^2=-\mu^2$. Here $g=\sqrt{2\lambda}$ and $m^2=2\lambda v^2$. Let ρ^{in} denote the in field which is the asymptotic limit of ρ. We have

$$(-\partial^2-m^2)\rho^{\text{in}}(x)=0. \tag{20}$$

In the following we consider the extended objects created by the condensation of ρ^{in}. The tree approximation leads to the following dynamical map for ρ:

$$\rho(x)=\rho^{\text{in}}(x)+\tfrac{3}{2}mg(-i)\int d^2y\Delta(x-y):[\rho^{\text{in}}(y)]^2:$$
$$+\left\{\tfrac{1}{2}g^2(-i)\int d^2y\Delta(x-y):[\rho^{\text{in}}(y)]^3:+\tfrac{9}{2}m^2g^2(-i)^2\int d^2yd^2z\Delta(x-y)\Delta(y-z):\rho^{\text{in}}(y)[\rho^{\text{in}}(z)]^2:\right\}+\cdots. \tag{21}$$

Here $\Delta(x-y)$ is the Green's function satisfying

$$(-\partial^2-m^2)\Delta(x-y)=i\delta^{(2)}(x-y). \tag{22}$$

We put (21) in the form

$$\rho(x)=\sum_{n=1}^{\infty}\rho^{(n)}(x), \tag{23}$$

where n denotes the order of the normal products. Then the following relation holds in the tree approximation:

$$\rho^{(n)}(x)=\tfrac{3}{2}mg(-i)\int d^2y\,\Delta(x-y):\sum_{i+j=n}\rho^{(i)}(y)\rho^{(j)}(y):+\tfrac{1}{2}g^2(-i)\int d^2y\,\Delta(x-y):\sum_{i+j+k=n}\rho^{(i)}(y)\rho^{(j)}(y)\rho^{(k)}(y):. \tag{24}$$

We now perform the boson transformation

$$\rho^{\text{ir.}}(x) \to \rho^{\text{in}}(x) + f(x)\,, \tag{25}$$

where the c-number function f satisfies

$$(-\partial^2 - m^2)f(x) = 0\,. \tag{26}$$

Denoting the boson-transformed ρ-field operator by ρ^f, we have

$$\phi_0^f(x) = v + \langle 0|\rho^f(x)|0\rangle\,. \tag{27}$$

We consider the static case. The space coordinates $x_1,\ y_1, \ldots$ will be simply denoted by $x,\ y, \ldots$. Then (26) and (27) read

$$\frac{d^2}{dx^2}f(x) = m^2 f(x) \tag{28}$$

and

$$\phi_0^f(x) = v + f(x) + \tfrac{3}{2}mg\int dy\,K(x-y)f^2(y) + \left[\tfrac{1}{2}g^2\int dy\,K(x-y)f^3(y) + \tfrac{9}{2}m^2g^2\int dy\,K(x-y)f(y)\int dz\,K(y-z)f^2(z)\right] + \cdots\,, \tag{29}$$

where the Green's function $K(x-y)$ is defined by

$$\left(\frac{d^2}{dx^2} - m^2\right)K(x-y) = \delta(x-y)\,. \tag{30}$$

The recurrence relation (24) becomes

$$\rho_f^{(n)}(x) = \tfrac{3}{2}mg\int dy\,K(x-y):\sum_{i+j=n}\rho_f^{(i)}(y)\rho_f^{(j)}(y): + \tfrac{1}{2}g^2\int dy\,K(x-y):\sum_{i+j+k=n}\rho_f^{(i)}(y)\rho_f^{(j)}(y)\rho_f^{(k)}(y):\,. \tag{31}$$

As a solution of (28), we choose $f(x)$ which diverges at $x=-\infty$ and regular at $x=\infty$; $f(x) = A\exp[-mx]$. Since $f(x)$ diverges at $x=-\infty$, we choose the Green's function K in such a way that $K(x-y)=0$ for $x>y$:

$$K(x) = -\theta(-x)\frac{1}{m}\sinh mx\,. \tag{32}$$

Noticing that $\rho_f^{(1)}(x) = f(x)$, we can easily see from (31) that

$$\rho_f^{(n)}(x) = C_n(e^{-mx})^n\,, \tag{33}$$

and C_n satisfy the recurrence formula

$$C_n = \frac{1}{m^2(n^2-1)}\left(\tfrac{3}{2}mg\sum_{i+j=n}C_iC_j + \tfrac{1}{2}g^2\sum_{i+j+k=n}C_iC_jC_k\right)\,,\quad (n\geq 2), \tag{34}$$

$$C_1 = A\,.$$

Solving (34), together with the relation $v = m/g$, we have

$$C_n = 2v\left(\frac{A}{2v}\right)^n\,, \tag{35}$$

which leads to

$$\phi_0^f(x) = v + 2v\sum_{n=1}^{\infty}\left(\frac{A}{2v}e^{-mx}\right)^n = v\,\frac{1+(A/2v)e^{-mx}}{1-(A/2v)e^{-mx}}\,. \tag{36}$$

When A is chosen as

$$A = -2ve^{ma}\,, \tag{37}$$

the $\phi_0^f(x)$ is obtained as

$$\phi_0^f(x) = v\tanh[(m/2)(x-a)]\,, \tag{38}$$

which is the well-known static solution of the Euler equation

$$(-\partial - \mu^2)\phi_0^f(x) = \lambda[\phi_0^f(x)]^3\,. \tag{39}$$

When $A = 2ve^{ma}$, $\phi_0^f(x)$ is given by

$$\phi_0^f(x) = v\coth[(m/2)(x-a)]\,, \tag{40}$$

which is another solution of (39).

For the static case, (38) and (40) are the only solutions which satisfy (39) with the condition $\phi_0^f(x)\to v(x\to\infty)$. The choice of $f(x)$ and the Green's function $K(x)$ is restricted from the well-defined property of the integration in the right-hand side of (29). If $f(x)$ is extended to be time dependent, there are many varieties for the choice of $f(x)$. The relation between the choice of $f(x)$ and the N-soliton solution will be a future problem.

The situation becomes involved when there appear certain bound states which prohibit us from using the successive iteration. Such a situation arises in many practical cases, in which the extended objects are created by condensation of composite Goldstone bosons (such as crystal phonons, magnons, etc.). When we cannot use the successive iteration, proof of the boson transformation theorem becomes very complicated. Since this has been discussed in Ref. 4, it will not be repeated here. In the following we present a sketch of the general structure of the boson method for derivation of the soliton solutions.

Suppose that the Heisenberg equation (1) gives rise to the set of in fields $(\varphi_1^{\text{in}}\varphi_2^{\text{in}}\cdots\varphi_n^{\text{in}}\cdots)$. Here $\varphi_\alpha^{\text{in}}$ $(\alpha=1,\ldots,n)$ are the boson in fields and other in fields are fermions. The free field equation for the boson in fields are

$$(-\partial^2-m_\alpha^2)\varphi_\alpha^{\text{in}}=0\,. \tag{41}$$

Since we require that ψ is realized in the Fock space of the in fields, all the matrix elements of ψ among the vectors in this Fock space should be well defined. This means that ψ can be expressed by a linear combination of normal products of the in fields:

$$\psi(x)=\psi(x;\varphi_\alpha^{\text{in}},\ldots)\,. \tag{42}$$

This is the dynamical map and should be understood as a weak relation. To calculate $F[\psi]$ in the right-hand side of the Heisenberg equation (1), we need to define the products of ψ. The products are defined by the rule which states that one first calculates the products of normal products of the in fields and rearranges them into a linear combination of normal products. The space-time integration should be performed only after the products of normal products are well taken care of. We assume that when certain divergences appear through the course of calculation, they can be eliminated by some regularization procedures in a reasonable manner. Once the products of ψ are determined, all the matrix elements of both sides of the Heisenberg equation (1) can be calculated. A well-known expression for the dynamical map is the Lehmann-Symanzik-Zimmermann (LSZ) formula.

Now perform the substitution called the boson transformation

$$\varphi_\alpha^{\text{in}}(x)\to\varphi_\alpha^{\text{in}}(x)+f_\alpha(x)\,, \tag{43}$$

in which $f_\alpha(x)$ $(\alpha=1,\ldots,n)$ are c-number functions which satisfy the equations for $\varphi_\alpha^{\text{in}}$,

$$(-\partial^2-m_\alpha^2)f_\alpha=0\,. \tag{44}$$

Then, the boson transformation theorem states that the boson-transformed Heisenberg field

$$\psi^f(x)=\psi(x;\varphi_\alpha^{\text{in}}+f_\alpha,\ldots) \tag{45}$$

also satisfies the Heisenberg equation (1), thus leading to (10). Note that the boson $\varphi_\alpha^{\text{in}}$ in (43) can be either elementary or composite.

The expression (43) for the boson transformation indicates that $f_\alpha(x)$ are created by the condensation of the bosons $\varphi_\alpha^{\text{in}}$. The result of this condensation is the appearance of certain extended objects. Since $f_\alpha(x)$ are c numbers, these extended objects behave classically: Intuitively speaking, the quantum fluctuation becomes much smaller than the macroscopic effect of the condensed bosons. The c-number field ϕ^f is defined by (11). Note that $\phi^f(x)$ describes a classically behaving extended object even when $\hbar\neq 0$. When we ignore $\hbar$, $\phi^f(x)$ becomes $\phi_0^f(x)$, i.e., Eq. (12). Following the argument which led to (13), we can prove that ϕ_0^f satisfies the classical Euler equation (15).

When the boson which makes the boson transformation happens to be composite, we should formulate the tree approximation in such a way that the internal lines of the tree diagrams include the propagation function of the composite boson. Such a formulation of the tree approximation can be made by means of the Ward-Takahashi relations. Since this kind of tree approximation has been formulated in Ref. 8 in which the itinerant electron ferromagnetism was studied, it will not be repeated here.

Equation (44) for f_α admits various propagating wave solutions. A well-known example of wavelike extended objects in solid state physics is the ultrasonic waves in crystals.

When we consider a static extended object, (44) becomes

$$(\nabla^2-m^2)f_\alpha=0\,. \tag{46}$$

When $m=0$, this equation has a trivial solution, i.e., $f_\alpha(x)=\text{constant}$. When we disregard this trivial solution, Eq. (46) with $m^2\geqslant 0$ does not admit any solution which is Fourier transformable. We thus conclude that the static objects created by the condensation of bosons carry certain singularities which prohibit the Fourier transform of f_α.

Since $f_\alpha(x)$ for static objects should carry certain singularities which prohibit its Fourier transform, $f_\alpha(x)$ has either a divergent singularity or a topological singularity. Here the divergent singularity means that $f_\alpha(x)$ diverges at $|\vec{x}|=\infty$ at least in certain directions of $\vec{x}$. The topological singularity means that $f_\alpha(x)$ is not single valued.

Since $f_\alpha(x)$ is path dependent when it is not single valued, the topological singularity can be mathematically expressed by the relation

$$G_{\mu\nu}^{\alpha\dagger}(x) \neq 0 \quad \text{for certain } x,\ \mu,\ \nu, \text{ and } \alpha\,, \tag{47}$$

where $G_{\mu\nu}^{\alpha\dagger}$ is defined by

$$G_{\mu\nu}^{\alpha\dagger}(x) \equiv [\partial_\mu, \partial_\nu] f_\alpha(x)\,. \tag{48}$$

Furthermore, existence of the path-dependent $f_\alpha(x)$ requires that $\partial_\rho f_\alpha(x)$ should be single valued,

$$[\partial_\mu, \partial_\nu]\partial_\rho f_\alpha(x) = 0\,. \tag{49}$$

Equation (44) together with (48) and (49) leads to

$$\partial_\nu f_\alpha(x) = \frac{1}{\partial^2 + m_\alpha^2}\,\partial^\mu G_{\mu\nu}^{\alpha\dagger}(x)\,, \tag{50}$$

where the derivative operator $(1/\partial^2 + m_\alpha^2)$ is defined in terms of the Fourier representation as follows:

$$\frac{1}{\partial^2 + m_\alpha^2}\, e^{ipx} = -\frac{1}{p^2 + m_\alpha^2}\, e^{ipx}\,. \tag{51}$$

Thus $(1/\partial^2 + m_\alpha^2)$ is the Green's function of the Klein-Gordon equation with the mass m_α. Note that $G_{\mu\nu}^{\alpha\dagger}$ is Fourier transformable. Equation (50) leads to $\partial^2 f_\alpha = 0$, implying that $m_\alpha = 0$. We thus conclude that the extended objects associated with the topological singularity (47) can be created only by the condensation of *massless* bosons. A systematic study of extended objects with topological singularities has been presented in Ref. 3.

We need a special remark when the Heisenberg field ψ is a fermion field, because $\langle 0|\psi^f|0\rangle$ vanishes. In this case we can still construct various bosonlike operators by means of products of an even number of ψ's. When we find a set of bosonlike operators [say $\varphi_a(\psi)$; $a = 1, \ldots, l$] which satisfy a closed set of Heisenberg equations, we can write down the classical equations of the same form. Latter classical equations are regarded as the Euler equations, and our consideration in this paper can be applied to these Euler equations. However, it frequently happens that the Heisenberg equations are not closed by a finite number of bosonlike operators; to obtain a closed set of Heisenberg equations for bosonlike operators, we usually need certain approximations. The Gor'kov equation[9] for the order parameter $\Delta(x)$ in the theory of superconductivity is a well-known example of this kind of Euler equation. There the Goldstone boson is a bound state of two electrons. When we denote the boson-transformed electron Heisenberg field by ψ^f, the order parameter $\Delta(x)$ is equal to $\langle 0|\psi_\downarrow^f\psi_\uparrow^f|0\rangle$. When $\Delta(x)$ is very small, the Gor'kov equation becomes the Ginzburg-Landau equation. When $\Delta(x)$ is not small, its Euler equation (i.e., the Gor'kov equation) has an extremely complicated structure. However, we can derive also a classical equation for physical quantities such as the electromagnetic field and current by applying the boson transformation to the dynamical maps of the electromagnetic Heisenberg field and electron Heisenberg field.[4,10] This equation can be used for the study of extended objects in superconductors. The same situation is true in the relativistic superconductorlike models (e.g., the Nambu model).

Let us close this paper with two comments on the boson method. In this paper we considered mostly $\phi^f(x)$ which is the vacuum expectation value of the boson-transformed Heisenberg field $\psi^f(x)$. When we calculate other matrix elements of ψ^f and the boson-transformed S matrix, we can study the reactions among extended objects and quantum particles. Another comment is the following: Since ϕ^f contains $\hbar$, it has quantum effects in it, although it describes a classical object. In other words, ϕ^f is the soliton solution with quantum correction; the quantum correction is given by the loop diagrams.

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